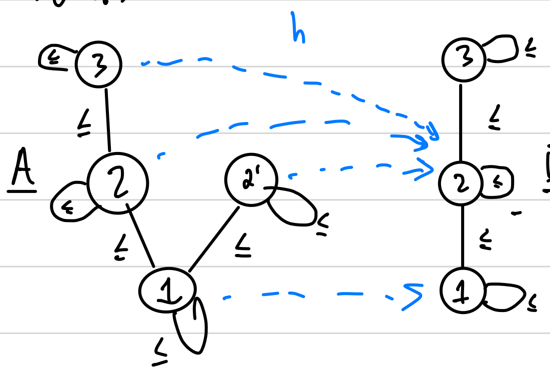
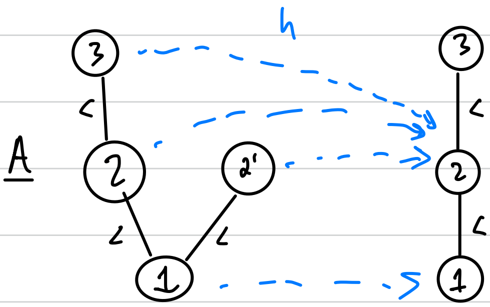
Math Logic: Model Theory & Computability
Lecture 04

Examples (continued). (c) Let $\underline{A} := (A, \leq)$ and $\underline{B} := (B, \leq)$ be the following partial orders:



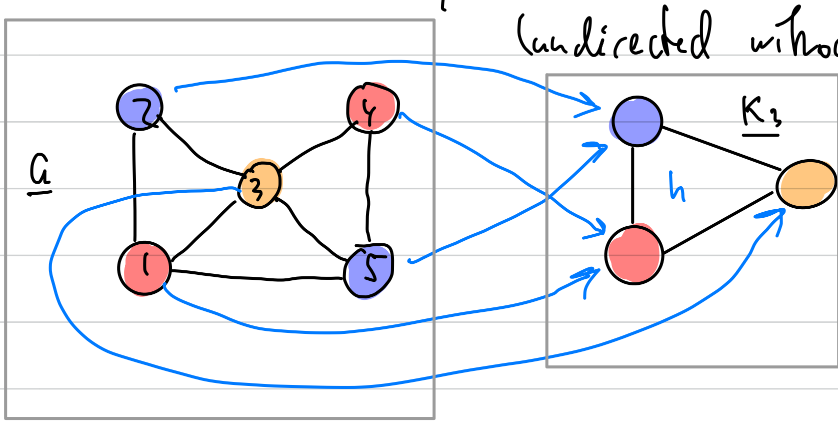
Then $h : A \to B$ is a $(\leq)$-homomorphism, in particular, we had $2 \leq 3$ in $\underline{A}$ and we still have $h(2) = 2 \leq 2 = h(3)$ in $\underline{B}$.

(d) Let $\underline{A} := (A, <)$ and $\underline{B} := (B, <)$ be the following strict partial orders:



Then h is not a $(<)$-homomorphism because $2 < 3$ in $\underline{A}$ but $h(2) = 2 \nless 2 = h(3)$ in $\underline{B}$.

(e) Note that a proper colouring of a graph $\underline{G} := (V, E)$ with n colours is equivalent to a homomorphism from $\underline{G}$ to the complete (undirected without loops) graph $\underline{K_n}$ on n vertices.



Indeed h is a homomorphism.

This implies that if $\underline{G}$ is n-colourable and $\underline{H} \to \underline{G}$ then $\underline{H}$ too is n-colourable. Conversely, if $\underline{H} \to \underline{G}$ and $\underline{H}$ is not n-colourable, then $\underline{G}$ too is not n-colourable.

Def. Let $\underline{A} := (A, \sigma)$, $\underline{B} := (B, \sigma)$ be σ-structures. A map $h : A \to B$ is called a *σ-isomorphism* if h is invertible ($\Leftrightarrow$ bijective) and both h and h^{-1} are σ-homomorphisms. $\underline{A}$ and $\underline{B}$ are called *isomorphic* if there is a σ-isomorphism $A \to B$ and we denote it $\underline{A} \cong \underline{B}$. We also write $h : \underline{A} \xrightarrow{\sim} \underline{B}$ to indicate that h is an isomorphism.

Obs. $h : A \to B$ being an isomorphism is equivalent to h being a bijective homomorphism with the additional property that

$$\vec{a} \in R^{\underline{A}} \iff h(\vec{a}) \in R^{\underline{B}}$$

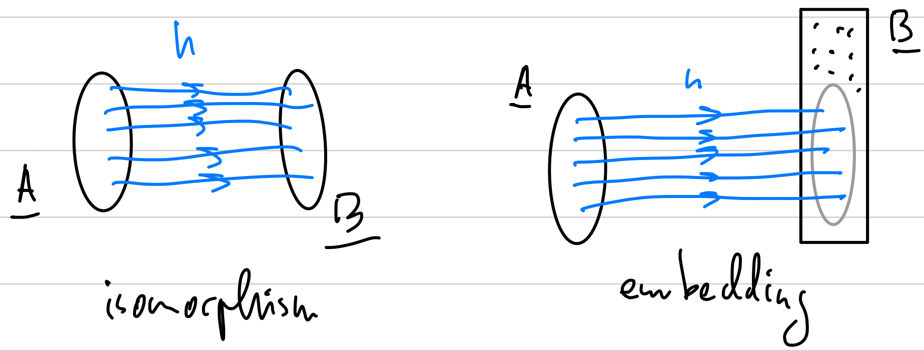
for each n-ary $R \in \mathrm{Rel}(\sigma)$ and $\vec{a} \in A^n$.

Def. Let $\underline{A} := (A, \sigma)$ and $\underline{B} := (B, \sigma)$ be σ-structures. A map $h : A \to B$ is called a *σ-embedding* if h is an isomorphism from $\underline{A}$ to the substructure $h(\underline{A})$ of $\underline{B}$, where $h(\underline{A})$ denotes the unique substructure supported by $h(A)$. In this case, we write $h : \underline{A} \hookrightarrow \underline{B}$ and we say that $\underline{A}$ *embeds into* $\underline{B}$, denoted by $\underline{A} \hookrightarrow \underline{B}$.

Obs. This is equiv. to h being an injective σ-hom. with the additional property that

$$\vec{a} \in R^{\underline{A}} \iff h(\vec{a}) \in R^{\underline{B}}$$

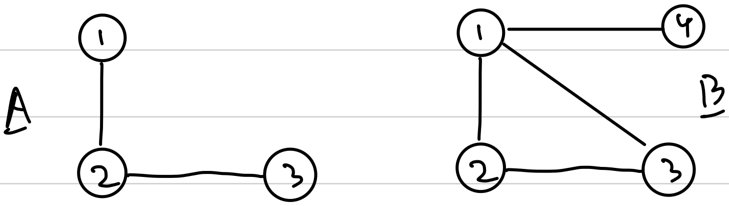
for each n-ary $R \in \mathrm{Rel}(\sigma)$ and $\vec{a} \in A^n$.



Examples. (a) $\underline{R} := (\mathbb{R}, 1, \cdot, ()^{-1})$ be the group of reals under $+$ in the signature $\sigma_{gp} := (1, \cdot, ()^{-1})$ where $1^{\underline{R}} := 0$, $\cdot^{\underline{R}} := +$, $(()^{-1})^{\underline{R}} := -()$.
Also let $\underline{R}^+ := (\mathbb{R}^+, 1, \cdot, ()^{-1})$, where $\mathbb{R}^+ := (0,\infty)$, $1^{\underline{R}^+} := 1$, $\cdot^{\underline{R}^+} := \cdot$, $(()^{-1})^{\underline{R}^+} := ()^{-1}$. Then
$$h : \mathbb{R} \to \mathbb{R}^+, \quad x \mapsto 2^x$$
is a σ_{gp}-isomorphism.

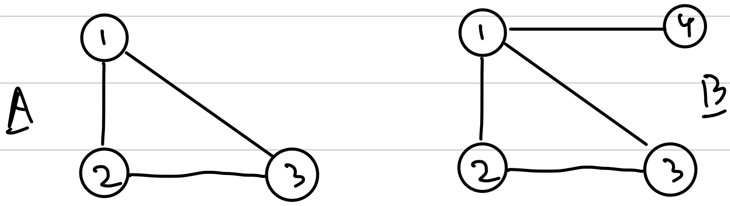
$\underline{R}^* := (\mathbb{R}\setminus\{0\}, 1, \cdot, ()^{-1})$, then
$$h : \mathbb{R} \to \mathbb{R}^*, \quad x \mapsto 2^x$$
is a σ_{gp}-embedding.

(b)



The inclusion map $\{1,2,3\} \to \{1,2,3,4\}$, $n \mapsto n$ is an injective homomorphism but not an embedding.

But below it is:



The language of first-order logic.

A signature σ contains names for constants, functions, and relations. Here we describe how to obtain new names for constants, function, and relations, allowing certain operations on them.

Def. For a signature σ, the (first-order) alphabet of σ, denoted A_σ, is the union of the following sets of symbols:

(i) $\mathrm{Const}(\sigma) \cup \mathrm{Funct}(\sigma) \cup \mathrm{Rel}(\sigma)$

(ii) Punctuation symbols: "(", ")", "," (comma)

(iii) Variables: $v_0, v_1, v_2, v_3, \dots$ (infinitely many)

(iv) logical connectives: $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation), $\rightarrow$ (implication), $\leftrightarrow$ (equivalence)

(v) Quantifiers: $\exists$ (exists), $\forall$ (for all).
([illegible])

We call finite sequences of symbols from A_σ **σ-words** or **words in A_σ**.

We first define names for new functions, called terms.

Def. A **σ-term** is a σ-word t obtained via the following inductive rules:

(i) $t := c$ is a σ-term for each $c \in \mathrm{Const}(\sigma)$.

(ii) $t := v_n$ is a σ-term for each variable v_n, i.e. each $n \in \mathbb{N}$.

(iii) $t := f(t_1, t_2, \dots, t_k)$ is a σ-term for each k-ary $f \in \mathrm{Funct}(\sigma)$ and σ-terms $t_1, t_2, \dots, t_k$.

Example. For $\sigma_{ring} := (0, 1, +, -(), \cdot)$, σ_{ring}-terms are exactly polynomials of several variables with integer coefficients. To see this, let's first decide that instead of writing $+(x,y)$ and $\cdot(x,y)$, we will write $(x+y)$ and $(x \cdot y)$. We also write x^k for $\underbrace{(\cdots((x \cdot x) \cdot x) \cdot \ldots) \cdot x}_{k \text{ times}}$. Lastly, we abbreviate $k \in \mathbb{N}^+$ as $\underbrace{(\cdots(1+1)+\ldots+)+1}_{k \text{ times}}$.
Then $((3 \cdot x^3) - (2 \cdot x \cdot y)) + 1$ is a σ_{ring}-term.

We now want to define interpretation of a σ-term in a σ-structure. The function symbols in σ come with a fixed arity. We would to be able to increase the arity. For example, in algebra, a polynomial $x^2 + x \cdot y + 1$ can be treated as a function of 2 or more variables; indeed, if we write $t := x^2 + x \cdot y + 1$, then $t(x, y, z)$ is a function of 3 variables.

Def. Let $\vec{v} := (v_{n_1}, v_{n_2}, \dots, v_{n_k})$ be a vector of distinct variables and let t be a σ-term. We call $t(\vec{v})$ an extended σ-term if all variables in t appear in $\vec{v}$.

Def. Let $\underline{A} := (A, \sigma)$ be a σ-structure and $t(\vec{v})$ be an extended σ-term, where $n := |\vec{v}|$. We define the interpretation of $t(\vec{v})$ in $\underline{A}$ as a function $t^{\underline{A}}(\vec{v}) : A^n \to A$ given by induction on the definition of t as follows:

(i) If $t := c$ for some $c \in Const(\sigma)$, then $t^{\underline{A}}(\vec{v})(\vec{a}) := c^{\underline{A}}$, i.e. $t^{\underline{A}}(\vec{v})$ is the constant $c^{\underline{A}}$ function on A^n.

(ii) If $t := v_k$ for some variable v_k, then v_k appears in $\vec{v} := (v_{e_1}, v_{e_2}, \dots, v_{e_n})$, i.e. $v_k = v_{e_m}$ for some m, and we define $t^{\underline{A}}(\vec{v})(a_1, a_2, \dots, a_n) := a_m$. In other words, $t^{\underline{A}}(\vec{v})$ is the projection onto the m^{th} coordinate on A^n.

(iii) If $t = f(t_1, t_2, \dots, t_k)$ for some k-ary $f \in Func(\sigma)$ and σ-terms $t_1, \dots, t_k$, then $t_1(\vec{v}), t_2(\vec{v}), \dots, t_k(\vec{v})$ are extended σ-terms so we can define $t^{\underline{A}}(\vec{v})(\vec{a}) := f^{\underline{A}}(t_1^{\underline{A}}(\vec{v})(\vec{a}), t_2^{\underline{A}}(\vec{v})(\vec{a}), \dots, t_k^{\underline{A}}(\vec{v})(\vec{a}))$.